Embedding Lemma

We now introduce a generalization of the triangle counting lemma. There are many versions and strenghenings of this lemma often called the counting lemma or embedding lemma.

Lemma 1 (Improved embedding lemma). *Let F be a k-chromatic graph with maximum degree $\Delta(F)$. Fix $0 < \delta < \frac{1}{k}$ and let G be a graph and let $V_1, \ldots, V_k$ be disjoint sets of vertices of G. If each V_i has $|V_i| \geq 2\delta^{-\Delta}|F|$ and each pair of partition classes is $\frac{1}{2\Delta}\delta^{\Delta}$-regular with density $\geq 2\delta$, then the classes $V_1, V_2, \ldots, V_k$ contain F.*

Proof. □

Lemma 2 ("Weak" embedding lemma). *Let F be a k-chromatic graph on f vertices ($2 \leq k \leq f$). Fix $0 < \delta < \frac{1}{k}$ and let G be a graph and let $V_1, \ldots, V_k$ be disjoint sets of vertices of G. If each V_i has $|V_i| \geq \delta^{-f}$ and each pair of partition classes is δ^f-regular with density $\geq 2\delta$, then the classes $V_1, V_2, \ldots, V_k$ contain F.*

Proof. We will prove the lemma with the two following strenghenings. First, we only require that the pairs of $V_1, \ldots, V_k$ have density at least $2\delta - \delta^f$. Second, we will show that if we (arbitrarily) label the color classes of F, then we can embed F into G such that the first color class of F is in V_1 and all other classes are in $V_2 \cup V_3 \cup \cdots V_k$.

We proceed by induction on f. For $f = 2$ the statement is trivial. Let $f > 2$ and assume the theorem holds for smaller values. Given F, let us remove a vertex y in the first color class to get $F' = F - y$.

For $2 \leq i \leq k$, let R_i be the set of vertices in V_1 that are adjacent to less than $\delta|V_i|$ vertices of V_i.

1: Show that $V_1 - \cup R_i$ is not empty. Hint: Use regularity of V_1, V_i to show that R_i is not too big.

Solution: If $|R_i| \geq \delta^f|V_1|$, then we have $|d(V_1, V_i) - d(R_i, V_i)| < \delta^f$ which implies that $d(V_1, V_i) < d(R_i, V_i) + \delta^f < \delta + \delta^f < 2\delta - \delta^f$; a contradiction. Thus, we have $|R_i| < \delta^f|V_1|$, so $|\cup R_i| < (k-1)\delta^f|V_1| < |V_1|$.

Thus, there is a vertex $x \in V_1 - \cup R_i$. Put $V_1' = V_1 - x$ and $V_i' = V_i \cap N(x)$ (for $i \neq 1$).

Now our goal is to show that $V_1', \ldots, V_k'$ satisfy the inductive hypothesis for $f - 1$.

2: First verify that V_i' is large enough for all i.

Solution: Observe that

$$|V_1'| = |V_1| - 1 \geq \delta|V_1| \geq \delta^{-(f-1)}$$

and for $i \neq 1$,

$$|V_i'| \geq \delta|V_i| \geq \delta^{-(f-1)}.$$

3: Check that V_i', V_j' is a regular pair for all i, j.

Solution: Now suppose $i \neq j$ and $A \subset V_i'$ and $B \subset V_j'$ are such that $|A| \geq \delta^{f-1}|V_i'| \geq \delta^f|V_i|$ and $|B| \geq \delta^{f-1}|V_j'| \geq \delta^f|V_j|$, then, with an application of the triangle inequality, we get

$$|d(V_i', V_j') - d(A, B)| \leq |d(V_i', V_j') - d(V_i, V_j)| + |d(V_i, V_j) - d(A, B)| < 2\delta^f < \delta^{f-1}.$$

4: Check that $d(V_i', V_j')$ is large enough for all i, j.

Solution: Finally,

$$|d(V_i, V_j) - d(V_i', V_j')| < \delta^f$$

implies that

$$d(V_i', V_j') > d(V_i, V_j) - \delta^f \geq 2\delta - 2\delta^f > 2\delta - \delta^{f-1}.$$

Thus $V_1', V_2', \ldots, V_k'$ satisfy the conditions of the theorem for $f-1$.

5: Use induction to finish the proof.

Solution: By induction we can embed F' into $G - x$ such that the first color class of F' is in V_1' and the other color classes of F' are in $V_2', \ldots, V_k'$. Because $x \in G$ is adjacent to every vertex in in $V_2', \ldots, V_k'$ we can put $y = x$ to embed F into G as desired.

□

We are now ready to give another proof of Erdő-Stone-Siminovits Theorem 3) using the regularity lemma and embedding lemma. Recall

Theorem 3. *Let F be a graph with chromatic number $\chi(F)$ and fix $0 < \delta < \frac{1}{\chi(F)}$, then there is an $n_0 = n_0(\delta, |F|)$ such that if G is a graph on $n \geq n_0$ vertices and*

$$e(G) > \left(1 - \frac{1}{\chi(F) - 1} + \delta\right) \frac{n^2}{2},$$

then G contains F as a subgraph.

Proof. Put $f = |F|$ and let G be as in the statement of the theorem. Let us apply the regularity lemma to G to with $\epsilon = (\frac{\delta}{8})^f$ and $m > \frac{8}{\delta} > \chi(F)$. That is, there is an equipartition $V_1, \ldots, V_r$ of G into r parts such that $\frac{8}{\delta} < r < M$ and all but at most $(\frac{\delta}{8})^f r^2$ of the pairs of clusters are $(\frac{\delta}{8})^f$-regular.

6: As in triangle removal lemma, count the number of edges 1. inside each cluster, 2. edges between clusters that are not regular, 3. edges between clusters with $d(V_i, V_j) < \frac{\delta}{4}$.

Solution: As in the triangle removal lemma we will remove the following edges.

1. Remove the edges inside of each cluster V_i. There are at most $r\binom{\lceil n/r \rceil}{2} \leq \frac{n^2}{r} < \frac{\delta}{8}n^2$ such edges.
2. Remove the edges between all pairs V_i, V_j that are not δ^f-regular. There are at most $2(\frac{\delta}{8})^f r^2$ such pairs and each has at most $(\frac{n}{r})^2$ edges. So we remove at most $(\frac{\delta}{8})^f n^2 \leq \frac{\delta}{8}n^2$ such edges
3. Remove the edges between all pairs V_i, V_j if the density of the pair $d(V_i, V_j) < \frac{\delta}{4}$. There are less than $\binom{r}{2}\frac{\delta}{4}(\frac{n}{r})^2 < \frac{\delta}{8}n^2$ such edges.

In total we have removed at less than $\delta\frac{n^2}{2}$ edges, so we still have more than

$$\left(1 - \frac{1}{\chi(H)-1}\right)\frac{n^2}{2}$$

edges.

7: Use Turán's theorem and the embedding lemma to finish the proof.

Solution: So, by Turán's theorem, the remaining graph contains a clique on $\chi(F)$ vertices. The $\chi(F)$ clusters that contain the vertices of this clique satisfy the conditions of the embedding lemma with $\frac{\delta}{8}$, so G contains F.

□

Now we go back and show application of the triangle removal lemma. Here we restate it for convenience.

Theorem 4 (Triangle removal lemma, Ruzsa-Szemerédi, 1978). *For $\alpha > 0$, there exists $\beta > 0$ such that if G is an n-vertex graph that requires the removal of αn^2 edges to be triangle-free, then G has at least βn^3 triangles.*

Recall that a k**-term arithmetic progression** (AP) is a sequence $a_1, a_2, \ldots, a_k$ such that there is some fixed d such that $a_{i+1} - a_i = d$ for all i.

Theorem 5 (Roth's theorem, 1953). *If $S \subset [n]$ contains no 3-term arithmetic progression, then $|S| = o(n)$.*

Proof. (Rusza-Szemerédi) Fix $\epsilon > 0$, and let $S \subset [n]$ of size ϵn. We will show that for all n large enough, that S contains a 3-term AP. We construct a graph G with vertex set partitioned into three classes $A = [n], B = [2n], C = [3n]$. If $s \in S$ and $x \in [n]$, then add to G the triangle between vertices $x \in A$, $x + s \in B$, and $x + 2s \in C$. Observe that for $b \in B$ and $c \in C$ forming an edge we necessarily have $c - b \in S$.

8: Use Triangle removal lemma to show that there are other triangles than just the ones we added. Hint: How many we added?

Solution: Clearly, the $n|S| = \epsilon n^2$ triangles formed in this way are edge-disjoint. To destroy these edge-disjoint triangles, we must remove ϵn^2 edges of G. Therefore, by the triangle removal lemma there are at least δn^3 triangles. For n large enough (depending on ϵ) we have $\delta n^3 > \epsilon n^2$ which implies that G contains a triangle other than those explicitly defined above.

9: Let x be the vertex of the triangle in A, that is not one of the explicitly added ones. Describe the triangle and use it to find a 3-term AP.

Solution: Because this triangle is different from those defined above we have distinct $b, c \in S$ such that $x + b \in B$ and $x + 2c \in C$ are the other vertices of the triangle. Thus $(x + 2c) - (x + b) = 2c - b$ is necessarily an element a of S. That is, $c - b = a - c$, i.e., b, c, a is a 3-term AP.

□

Theorem 6 (Solymosi, 2001)**.** *For any $\alpha > 0$, there exists N such that if A is a set of $\geq \alpha N^2$ many points on the $N \times N$ integer lattice, then A contains three distinct points of the following form $(x, y), (x + d, y), (x, y + d)$, i.e., an isosceles right triangle.*

Proof. Consider the collection of vertical lines, horizontal lines and 45° diagonal (left to right) lines in the $N \times N$ lattice.

Construct a 3-partite graph G with classes X, Y, Z such that X is the set of vertical lines, Y is the set of horizontal lines, and Z is the set of diagonal lines.

Two vertices (lines) in this graph are connected by an edge if the intersection of the two lines is an element of A. Therefore, for each element $a \in A$, the three lines intersecting in a form a triangle in G.

10: Finish the proof by looking at triangles. □